

Compact forms of Complex Lie Supergroups

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Abstract

In this paper we construct compact forms associated with a complex Lie supergroup with Lie superalgebra of classical type.

1 Introduction

In the ordinary theory of semisimple Lie groups, we can associate to any semisimple complex Lie group a corresponding real compact form. It is then very natural to ask, whether the same statement is true in the context of Lie supergroups. Given the rigidity of supergeometry requirements, it turns out that not all of the Lie supergroups with Lie superalgebras of classical type admit such a compact form, however under some hypothesis it is possible to grant such existence.

Our starting point is a complex analytic supergroup G with $\mathfrak{g} = \text{Lie}(G)$ of classical type. Such G corresponds to a unique affine algebraic supergroup constructed via the Chevalley recipe as detailed in [9], [10]. Once a suitable real form \mathfrak{k} of \mathfrak{g} is obtained through an involution, we proceed and build the corresponding compact supergroup K , with \mathfrak{k} as its Lie superalgebra. The superalgebra of global sections of K is precisely given by the global sections of G invariant under the involution defined through the involution of \mathfrak{g} defining \mathfrak{k} . We repeat the same construction using the Super Harish-Chandra Pairs (SHCP) terminology establishing a complete equivalence between the two different approaches to this problem.

Acknowledgements. We wish to thank prof. V. S. Varadarajan and prof. C. Carmeli for helpful comments. We are also indebted to our referee for very valuable suggestions.

2 Preliminaries

We review few facts about supergeometry, sending the reader to [5, 16] for more details.

Let $k = \mathbf{R}$ or \mathbf{C} be our ground field.

A *superalgebra* A is a \mathbf{Z}_2 -graded algebra, $A = A_0 \oplus A_1$, where $p(x)$ denotes the parity of an homogeneous element x . The superalgebra A is said to be *commutative* if for any two homogeneous elements x, y , $xy = (-1)^{p(x)p(y)}yx$. All of our superalgebras are assumed to be commutative unless otherwise specified and their category will be denoted by $(\text{salg})_k$.

Definition 2.1. A *superspace* $S = (|S|, \mathcal{O}_S)$ is a topological space $|S|$ endowed with a sheaf of superalgebras \mathcal{O}_S such that the stalk at a point $x \in |S|$ denoted by $\mathcal{O}_{S,x}$ is a local superalgebra for all $x \in |S|$, i. e. it has a unique (two-sided) ideal. A *morphism* $\phi : S \rightarrow T$ of superspaces is given by $\phi = (|\phi|, \phi^*)$, where $|\phi| : |S| \rightarrow |T|$ is a map of topological spaces and $\phi^* : \mathcal{O}_T \rightarrow \phi_* \mathcal{O}_S$ is a local sheaf morphism, that is, $\phi_x^*(\mathbf{m}_{|\phi|(x)}) = \mathbf{m}_x$, where $\mathbf{m}_{|\phi|(x)}$ and \mathbf{m}_x are the maximal ideals in the stalks $\mathcal{O}_{T,|\phi|(x)}$ and $\mathcal{O}_{S,x}$ respectively.

If \mathcal{I}_S is the ideal sheaf generated by the odd sections in \mathcal{O}_S , we define the ordinary ringed space $S_0 = (|S|, \mathcal{O}_S/\mathcal{I}_S)$ as the *ringed space underlying* the superspace S .

The most important examples of superspaces are given by *supermanifolds* and *superschemes*.

Example 2.2. 1. Let $k = \mathbf{R}$ or \mathbf{C} . The superspace $k^{p|q}$ is the topological space k^p endowed with the following sheaf of superalgebras. For any $U \subset_{\text{open}} k^p$

$$\mathcal{O}_{k^{p|q}}(U) = \mathcal{O}_{k^p}(U) \otimes k[\xi_1, \dots, \xi_q],$$

where $k[\xi_1, \dots, \xi_q]$ is the exterior algebra generated by the q variables ξ_1, \dots, ξ_q and \mathcal{O}_{k^p} denotes the ordinary C^∞ or analytic sheaf over k^p .

2. *SpecA.* Let A be an object of $(\text{salg})_k$. Since A_0 is an algebra, we can consider the topological space

$$\text{Spec}(A_0) = \{\text{prime ideals } \mathbf{p} \subset A_0\}$$

with its structural sheaf \mathcal{O}_{A_0} . The stalk $A_{\mathbf{p}}$ of the structural sheaf at the prime $\mathbf{p} \in \text{Spec}(A_0)$ is the localization of A_0 at \mathbf{p} . As for any superalgebra, A is a module over A_0 , and we have a sheaf \tilde{A} of \mathcal{O}_{A_0} -modules over $\text{Spec} A_0$ with stalk $A_{\mathbf{p}}$, the localization of the A_0 -module A over each prime $\mathbf{p} \in \text{Spec}(A_0)$. $\underline{\text{Spec}} A =_{\text{def}} (\text{Spec} A_0, \tilde{A})$ is a superspace.

Definition 2.3. A *differentiable* (resp. *analytic*) *supermanifold* of dimension $p|q$ is a superspace $M = (|M|, \mathcal{O}_M)$ which is locally isomorphic to the superspace $k^{p|q}$, where the ordinary sheaf over k^p is taken to be the C^∞ or the analytic sheaf respectively in case of $k = \mathbf{R}$ or $k = \mathbf{C}$.

An *affine superscheme* is a superspace isomorphic to $\underline{\text{Spec}} A$ for a superalgebra A . A *superscheme* is a superspace which is locally isomorphic to affine superschemes.

Morphisms of supermanifolds or of superschemes are just the morphism of the corresponding superspaces.

Example 2.4. 1. *Affine superspace.* Define

$$\mathbf{A}^{m|n} := \underline{\text{Spec}} k[x_1, \dots, x_m, \xi_1, \dots, \xi_n]$$

(the odd variables $\xi_1 \dots \xi_n$ anticommute). This superscheme is called the *affine superspace* of dimension $m|n$. Its underlying scheme is the affine space \mathbf{A}^m of dimension m .

2. *General linear supergroup.* Let $V = V_0 \oplus V_1$ be a finite dimensional vector superspace. Define

$$\text{GL}(V) = (|\text{GL}(V_0)| \times |\text{GL}(V_1)|, \mathcal{O}_{\text{End}(V)}|_{|\text{GL}(V_0)| \times |\text{GL}(V_1)|})$$

where $\text{End}(V)$ is the super vector space of endomorphism of V and $\mathcal{O}_{\text{End}(V)}$ denotes the supersheaf of differential (resp. analytic or algebraic) sections on $\text{End}(V)$. Since $|\text{GL}(V_0)| \times |\text{GL}(V_1)|$ is open in $|\text{End}(V)|$ we have a superspace, which is a supermanifold or an affine superscheme resp., depending on whether we take $\text{End}(V)$ as in Ex. 2.2 (1) or as in Ex. 2.4, (1).

Next we want to introduce the concept of *functor of points* of a superscheme (resp. supermanifold).

Definition 2.5. The *functor of points* of a superscheme X is a representable functor from the category of superschemes to the category of sets:

$$\begin{aligned} h_X : (\text{sschemes})^o &\longrightarrow (\text{sets}) \\ T &\longrightarrow h_X(T) = \text{Hom}(T, X) \end{aligned}$$

and $h_X(\phi)f = f \circ \phi$ for any morphism $\phi : T \longrightarrow S$. The elements in $h_X(T)$ are called the *T-points* of X . (The label ‘ o ’ means that we are taking the opposite category.)

In the very same way it is possible to define the functor of points of a supermanifold.

It is customary to use the same letter to denote a supermanifold (or a superscheme) and its functor of points. We shall use two different symbols X and h_X only when strictly necessary.

Observation 2.6. The functor of points of a superscheme is determined by its restriction to the category of affine superschemes, which is equivalent to the category of affine superalgebras : $(\text{salg})_k$ (see [5] ch. 10). Hence given a superscheme X , we can equivalently define its functor of points as:

$$\begin{aligned} h_X : (\text{salg})_k &\longrightarrow (\text{sets}) \\ A &\longrightarrow h_X(A) = \text{Hom}(\underline{\text{Spec}}A, X) \end{aligned}$$

and $h_X(\phi)f = f \circ \underline{\text{Spec}}\phi$ for any morphism $\phi : A \longrightarrow B$. The elements in $h_X(A)$ are called the *A-points* of X . When X is itself an affine superscheme, its functor of points is *representable*:

$$\begin{aligned} h_X : (\text{salg})_k &\longrightarrow (\text{sets}) \\ A &\longrightarrow h_X(A) = \text{Hom}(\mathcal{O}(X), A) \end{aligned}$$

and $h_X(\phi)f = \phi \circ f$ for any morphism $\phi : A \longrightarrow B$. $\mathcal{O}(X)$ denotes the superalgebra of global section of \mathcal{O}_X . The functor of points of a superscheme $F : (\text{salg})_k \longrightarrow (\text{sets})$ is a local functor i. e. it has the sheaf property. In other words let $A \in (\text{salg})_k$ and $(f_i, i \in I) = (1) = A$. Let $\phi_k : A \longrightarrow A_{f_k}$ be the natural map of the algebra A to its localization and also $\phi_{kl} : A_{f_k} \longrightarrow A_{f_k f_l}$. Then, for a family $\alpha_i \in F(A_{f_i})$, such that $F(\phi_{ij})(\alpha_i) = F(\phi_{ji})(\alpha_j)$ there exists $\alpha \in F(A)$ such that $F(\phi_i)(\alpha) = \alpha_i$. A generic functor $F : (\text{salg})_k \longrightarrow$

(sets) is the functor of points of a superscheme if and only if it is local and it admits a cover by open affine subfunctors (i.e. the functor of points of an affine superscheme) (ref. [5] ch. 10, Representability Criterion).

Finally we recall that by Yoneda's lemma, given two superschemes (resp. supermanifolds) S and T , the natural transformations $h_S \rightarrow h_T$ are in one-to-one correspondence with the superscheme (resp. supermanifold) morphisms $S \rightarrow T$. Consequently two superschemes (resp. supermanifolds) are isomorphic if and only if their functor of points are isomorphic.

3 Analytic and Algebraic Supergroups

A supermanifold (resp. affine superscheme) G is a *supergroup* when its functor of points h_G is valued in the category of groups; this is equivalent to say that the superalgebra of global sections $\mathcal{O}(G)$ has a Hopf superalgebra structure (see [5, 16] for more details on Lie, analytic and algebraic supergroups). We say that the supergroup scheme G is an *algebraic supergroup* when G_0 is algebraic and $\mathcal{O}(G)$ is finitely generated as $\mathcal{O}(G)_0$ -module.

A supergroup G is called a *matrix supergroup* if G admits an embedding into $\mathrm{GL}(V)$, where we fix a homogeneous basis for the super vector space V .

From now on all of our supergroups are assumed to be matrix supergroups.

We now want to establish a correspondence between complex analytic Lie supergroups and algebraic supergroup schemes. The next observation is crucial in the developing of our work.

Observation 3.1. If G is an affine algebraic complex supergroup, G has a unique analytic supergroup structure. This happens in the same way as in the ordinary setting and the details of the construction appear in [8]. Vice-versa if $G = (|G|, \mathcal{O}_G)$ is a complex analytic matrix supergroup (i.e. G admits an embedding into some $\mathrm{GL}(V)$), there exists a unique algebraic complex supergroup G_a associated with it. In fact the sheaf of superalgebras \mathcal{O}_G splits, that is $\mathcal{O}_G \cong \mathcal{O}_{G_{\mathrm{red}}} \otimes \wedge(\xi_1 \dots \xi_q)$, where G_{red} denotes the ordinary analytic group underlying G . Since G_{red} has a unique algebraic group $G_{a,\mathrm{red}}$ associated with it, we can define the sheaf of G_a as $\mathcal{O}_{G_a} = \mathcal{O}_{G_{a,\mathrm{red}}} \otimes \wedge(\xi_1 \dots \xi_q)$ and $G_a = \underline{\mathrm{Spec}}\mathcal{O}(G_a)$.

Hence we can view the same supergroup G as an analytic or an algebraic supergroup. We shall use the same letter to denote both, when the context makes clear whether we are considering the analytic or the algebraic structure on $|G|$.

Assume now that G is a complex Lie supergroup, with Lie superalgebra \mathfrak{g} of classical type, that is \mathfrak{g} is non abelian simple (no nontrivial ideals) and \mathfrak{g}_1 is completely reducible as \mathfrak{g}_0 -module. All of what we say holds also for G with Lie superalgebra which is a direct sum of superalgebras of classical type. We shall nevertheless, for clarity of exposition, examine only the case of \mathfrak{g} of classical type. According to [12] we have a complete list of such Lie superalgebras:

Theorem 3.2. *The complex Lie superalgebras of classical type are either isomorphic to a simple Lie algebra or to one of the following Lie superalgebras:*

$$A(m, n), \quad m \geq n \geq 0, \quad m + n > 0; \quad B(m, n), \quad m \geq 0, n \geq 1; \quad C(n), \quad n \geq 3$$

$$D(m, n), \quad m \geq 2, n \geq 1; \quad P(n), \quad n \geq 2; \quad Q(n), \quad n \geq 2$$

$$F(4); \quad G(3); \quad D(2, 1; a), \quad a \in k \setminus \{0, -1\}.$$

Any complex affine algebraic supergroup with Lie superalgebra of classical type can be built via the Chevalley's supergroups recipe (ref. [10]). Briefly, this amounts to choose a suitable complex representation V of the complex Lie superalgebra \mathfrak{g} of classical type with root system Δ . If $(\text{salg})_k$ denotes the category of commutative superalgebras, the functor of points of the Chevalley supergroup $G : (\text{salg})_k \longrightarrow (\text{sets})$ is defined as:

$$G(A) = \langle G_0(A), 1 + \theta_\alpha X_\alpha, \alpha \in \Delta_1, \theta_\alpha \in A_1 \rangle \subset \text{GL}(V)(A), \quad A \in (\text{salg})_k$$

where as usual we omit the representation and write just X_α for the action of the root vector X_α on V in a fixed Chevalley basis (see [9] for the definition of Chevalley basis in the super context). Since $G(A) \subset \text{GL}(V)(A)$, the definition on the morphism is clear.

In [9] we prove that such a functor is representable, so that it is indeed an algebraic supergroup.

4 Compact form of a Lie superalgebra

Assume G is a complex analytic Lie supergroup, \mathfrak{g} its Lie superalgebra, \mathfrak{g} of classical type.

Definition 4.1. We say that the real Lie superalgebra $\mathfrak{k} \subset \mathfrak{g}$ is a *compact real form* of \mathfrak{g} if

- \mathfrak{k}_0 is of compact type (i. e. $\text{Ad}(\mathfrak{k}_0)$ is compact);
- $\mathbb{C} \otimes_{\mathbb{R}} \mathfrak{k} = \mathfrak{g}$.

We say that a real Lie supergroup K is *maximal compact* in G if $\text{Lie}(K)$ is a compact real form of \mathfrak{g} .

As we will see in Obs. 4.3, contrary to what happens in the ordinary setting, the existence of compact real forms is not granted for \mathfrak{g} of every classical type.

We can determine a compact real form of a Lie superalgebra of classical type via an involution. Let \mathfrak{h} be a CSA of \mathfrak{g} , $\Delta = \Delta^+ \cup \Delta^-$ the root system. Let $\{H_j, X_\alpha\}$ be a Chevalley basis for \mathfrak{g} ; the existence of such basis is granted for \mathfrak{g} , see [9]. Let $H_\alpha = [X_\alpha, X_{-\alpha}]$, this is an integral combination of the H_j 's (see [9]).

Assume

$$[X_\alpha, X_\alpha] = 0, \quad \text{for all } \alpha \in \Delta \quad (1)$$

Theorem 4.2. *Let \mathfrak{g} be as above and let:*

$$\mathfrak{k} := \text{span}_{\mathbb{R}}\{iH_j, i(X_\alpha + X_{-\alpha}), (X_\alpha - X_{-\alpha})\} \subset \mathfrak{g}$$

Then \mathfrak{k} is a compact real form of \mathfrak{g} , moreover \mathfrak{k} consists of the fixed points of the involutory semiautomorphism:

$$\begin{array}{ccc} s : & \mathfrak{g} & \longrightarrow \mathfrak{g} \\ & X_\alpha & \mapsto -X_{-\alpha} \\ & H_j & \mapsto -H_j \end{array}$$

Proof. \mathfrak{k} is a real subalgebra of \mathfrak{g} , it is fixed by s and it is a compact real form of \mathfrak{g} , by the ordinary theory. ■

Observation 4.3. If $[X_\alpha, X_\alpha] \neq 0$, one can check right away that \mathfrak{k} written above is not closed for the bracket, so the semiautomorphism s makes no sense in this context. Root vectors X_α with such property exist for example for $\mathfrak{osp}(V)$. In this case there is no compact real form of \mathfrak{g} . In fact let us look at $\mathfrak{osp}(1|2)$. The even part $\mathfrak{osp}(1|2)_0 = \mathfrak{sl}_2(\mathbf{C})$ acts on the odd part: $\mathfrak{osp}(1|2)_1 = \mathbf{C}^2 \oplus (\mathbf{C}^2)^*$, via the standard and contragredient representation of $\mathfrak{sl}_2(\mathbf{C})$. The compact real form of $\mathfrak{sl}_2(\mathbf{C})$ is $\mathfrak{su}(2)$, which does not admit even dimensional real representations that complexified give the $\mathfrak{osp}(1|2)_0$ -module $\mathfrak{osp}(1|2)_1$. This example shows that our hypothesis $[X_\alpha, X_\alpha] \neq 0$ leaves out some superalgebras that do not admit compact real forms, among which the families of Lie superalgebras of classical type: B , C , D . A similar argument excludes also the strange superalgebra $Q(2)$.

The existence of a compact real form \mathfrak{k} of \mathfrak{g} under our hypotheses allows us to proceed into the construction of the corresponding supergroup K , which will be compact, since its underlying topological space is compact and maximal, in the sense that the complexification of its Lie superalgebra coincides with \mathfrak{g} . We shall do it in two different ways: via an algebraic method using the Chevalley supergroups theory and via the Super Harish Chandra Pairs theory. In the end, we will give a comparison between these two different, but equivalent approaches.

5 Compact subgroups

Let G be a complex analytic Lie supergroup, with Lie superalgebra \mathfrak{g} of classical type, subject to hypothesis (1). We want to introduce G^r the real supergroup underlying G . The question of real forms is treated in [6] with different language, but we shall nevertheless be inspired by the same philosophy.

As before, let G denote both the complex analytic supergroup and the unique algebraic supergroup associated with it (see sec. 3).

The functor of points of the complex affine algebraic supergroup G associates a group to each object in $(\text{salg})_{\mathbf{C}}$, the category of complex commutative superalgebras:

$$G : (\text{salg})_{\mathbf{C}} \longrightarrow (\text{groups}), \quad A \longmapsto G(A) = \text{Hom}(\mathcal{O}(G), A)$$

where $\mathcal{O}(G)$ is the superalgebra of global sections of the structural sheaf of G . Since all the supergroups we are interested in are subgroups of $\mathrm{GL}(V)$ for a suitable superspace V , the functor G is automatically defined on the morphisms. Similarly the functor of points of a real affine algebraic supergroup associates to any *real* commutative superalgebra a group. Now we want to understand what is the real algebraic supergroup G^r underlying G .

Definition 5.1. Define the following functor:

$$G^r : (\mathrm{salg})_{\mathbf{R}} \longrightarrow (\mathrm{groups}), \quad A \longmapsto G^r(A) = \mathrm{Hom}_{(\mathrm{salg})_{\mathbf{C}}}(\mathcal{O}(G), A \otimes_{\mathbf{R}} \mathbf{C})$$

where $(\mathrm{salg})_{\mathbf{R}}$ is the category of commutative \mathbf{R} -superalgebras. We say that G^r is the *real algebraic supergroup underlying the complex supergroup G* .

In [9] and [5] ch. 7, we showed that the homogeneous one-parameter subgroups of G come in three different fashions. Let $X_\alpha, X_\beta, X_\gamma$ be root vectors in the Chevalley basis, $\alpha \in \Delta_0, \beta, \gamma \in \Delta_1$, with $[X_\beta, X_\beta] = 0, [X_\gamma, X_\gamma] \neq 0$. We define the following supergroup functors from the categories of complex commutative superalgebras to the category of sets:

$$\begin{aligned} x_\alpha(A) &:= \{x_\alpha(t) := \exp(t X_\alpha) \mid t \in A_0\} = \\ &= \left\{ \left(1 + t X_\alpha + t^2 \frac{X_\alpha^2}{2} + \dots\right) \mid t \in A_0 \right\} \subset \mathrm{GL}(V)(A), \\ x_\beta(A) &:= \{x_\beta(\vartheta) := \exp(\vartheta X_\beta) \mid \vartheta \in A_1\} \\ &= \{ (1 + \vartheta X_\beta) \mid \vartheta \in A_1 \} \subset \mathrm{GL}(V)(A), \\ x_\gamma(A) &:= \{x_\gamma(t, \theta) := \exp(\vartheta X_\gamma + t X_\gamma^2) \mid \vartheta \in A_1, t \in A_0\} = \\ &= \{ (1 + \vartheta X_\gamma) \exp(t X_\gamma^2) \mid \vartheta \in A_1, t \in A_0 \} \subset \mathrm{GL}(V)(A). \end{aligned}$$

These functors are all representable and their representing Hopf superalgebras are respectively: $\mathbf{C}[T], \mathbf{C}[\Theta], \mathbf{C}[T, \Theta]$. The Hopf structure of the first two is trivial, while for the last one see [9]. Notice that x_γ will not appear in our G , since our given hypothesis (1).

We want to understand what are the real supergroups underlying x_α, x_β . Let us consider the case β odd, $[X_\beta, X_\beta] = 0$ (the case x_α, α even is the

same).

$$x_\beta^r : (\text{salg})_{\mathbf{R}} \longrightarrow (\text{sets}), \quad x_\beta^r(A) = \text{Hom}(\mathbf{C}[\Theta], A \otimes_{\mathbf{R}} \mathbf{C})$$

Hence an element in $x_\beta^r(A)$ is a morphism:

$$\mathbf{C}[\Theta] \longrightarrow A \otimes_{\mathbf{R}} \mathbf{C}, \quad \Theta \longmapsto \theta := \theta_0 \otimes 1 + \theta_1 \otimes i$$

We denote such morphism $x_\beta^r(\theta)$ to stress the fact that it is determined once we choose θ a pair of elements $\theta_0, \theta_1 \in A_1$ the odd part of the real commutative superalgebra A . It makes then perfect sense to consider the element $x_\beta^r(\bar{\theta})$:

$$\mathbf{C}[\Theta] \longrightarrow A \otimes_{\mathbf{R}} \mathbf{C}, \quad \Theta \longmapsto \bar{\theta} := \theta_0 \otimes 1 - \theta_1 \otimes i$$

where $\bar{\theta}$ is now associated to the pair of odd elements $(\theta_0, -\theta_1)$.

We are ready for one of our main definitions.

Definition 5.2. Let the notation be as above. Define the following natural involution:

$$\sigma_A : G^r(A) \longrightarrow G^r(A)$$

$$x_\beta(\theta) \mapsto x_{-\beta}(-\bar{\theta})$$

$$g \mapsto \sigma_A^0(g), \quad g \in G_0^r(A)$$

where $\sigma^0 : G_0^r \longrightarrow G_0^r$ is the involution of the reduced group G_0^r detailed in [15].

Remark 5.3. The ordinary involution σ_A^0 of $G_0^r(A)$ is such that $\sigma_A^0(x_\alpha(t)) = x_{-\alpha}(-\bar{t})$, for α an even root and $\sigma_A^0(h(t)) = h(\bar{t}^{-1})$ for h a toral element (see [15]). We cannot however define σ_A^0 by just specifying the images of $x_\alpha(t)$, $h(t)$ only, since these elements generate $G_0^r(A)$ only for A local (see [7]). The existence of such σ^0 is however granted by the ordinary theory. We do not incur into the same problem in the super setting, since $G^r(A)$ is indeed generated by $G_0^r(A)$ and $x_\beta(A)$ for all superalgebras A (see [9]).

We now wish to relate the two involutions σ_A on $G^r(A)$ (see Def. 5.2) and s on \mathfrak{g} (see Theorem 4.2) and show they correspond the same (semi) automorphism of $\mathcal{O}(G)$, the Hopf superalgebra representing the supergroup G with Lie superalgebra \mathfrak{g} . We are deeply indebted to our referee for his suggestions regarding the next observation.

Observation 5.4. Let V be a complex super vector space (resp. a Lie superalgebra or an Hopf superalgebra). We define \overline{V} as the complex super vector space where $c, v \mapsto cv$ is replaced by $c, v \mapsto \bar{c}v$ for $c \in \mathbf{C}$ (and similarly in the case of V a Lie or Hopf superalgebra). Giving a \mathbf{C} -semilinear involution on V is the same as giving a \mathbf{C} -linear isomorphism $V \rightarrow \overline{V}$.

Consider now the \mathbf{C} -semilinear involution $s : \mathfrak{g} \rightarrow \mathfrak{g}$ as in 4.2. Such an involution corresponds to a \mathbf{C} -linear isomorphism $\hat{s} : \mathfrak{g} \rightarrow \overline{\mathfrak{g}}$, which extends uniquely to give an isomorphism $\mathcal{U}(\hat{s}) : \mathcal{U}(\mathfrak{g}) \rightarrow \mathcal{U}(\overline{\mathfrak{g}})$. We wish to show that such \hat{s} corresponds to a unique Hopf superalgebra isomorphism $\sigma' : \overline{\mathcal{O}(G)} \rightarrow \mathcal{O}(G)$ (equivalently to the \mathbf{C} -semilinear involution $\sigma' : \mathcal{O}(G) \rightarrow \mathcal{O}(G)$) inducing such \hat{s} on the Lie superalgebras.

We need to go back briefly to the construction of Chevalley supergroups (Ref. [9]). By our hypothesis G is constructed via the Chevalley supergroups recipe, in other words $G = G_V$, where $\rho : \mathfrak{g} \rightarrow \text{End}_{\mathbf{C}}(V)$ is a faithful representation of \mathfrak{g} into some complex super vector space V . In the above notation, we can define immediately $\bar{\rho} : \overline{\mathfrak{g}} \rightarrow \overline{\text{End}_{\mathbf{C}}(V)} = \text{End}_{\mathbf{C}}(\overline{V})$, hence we obtain another algebraic supergroup $G_{\overline{V}}$, through the Chevalley supergroup construction. Since all of our objects are defined over the reals, we have that $G_{\overline{V}}$ is represented by the Hopf superalgebra $\overline{\mathcal{O}(G)}$. So, in order to obtain the desired Hopf superalgebra isomorphism $\sigma' : \overline{\mathcal{O}(G)} \rightarrow \mathcal{O}(G)$, it is enough that we give an isomorphism $G_V \cong G_{\overline{V}}$. For this we consider the faithful representation $\rho \cdot \hat{s} : \mathfrak{g} \rightarrow \text{End}_{\mathbf{C}}(\overline{V})$. Both ρ and $\rho \cdot \hat{s}$ have the same weights (up to sign), hence according to the recipe, they will give an isomorphism $\psi : G_V \xrightarrow{\sim} G_{\overline{V}}$, ($\psi = \underline{\text{Spec}}(\sigma')$).

This implies the isomorphism

$$\psi_A : G_V(A \otimes_{\mathbf{R}} \mathbf{C}) \xrightarrow{\sim} G_{\overline{V}}(A \otimes_{\mathbf{R}} \mathbf{C}), \quad \forall A \in (\text{salg})_{\mathbf{R}}$$

Consider now the isomorphism $c : G_{\overline{V}}(R) \cong G_V(\overline{R})$ obtained via the complex conjugation. Then we have:

$$\sigma_A : G_V(A \otimes_{\mathbf{R}} \mathbf{C}) \xrightarrow{\underline{\text{Spec}}(\sigma')} G_{\overline{V}}(A \otimes_{\mathbf{R}} \mathbf{C}) \xrightarrow{c} G_V(A \otimes_{\mathbf{R}} \mathbf{C}), \quad A \in (\text{salg})_{\mathbf{R}}$$

(since $A \in (\text{salg})_{\mathbf{R}}$ we have $\overline{A \otimes_{\mathbf{R}} \mathbf{C}} = A \otimes_{\mathbf{R}} \mathbf{C}$). Notice that in our context $G^r(A) = G_V(A \otimes_{\mathbf{R}} \mathbf{C}) = \text{Hom}_{(\text{salg})_{\mathbf{C}}}(\mathcal{O}(G), A \otimes \mathbf{C})$.

We now come to the uniqueness of σ' . Since G is connected, we have that $\mathcal{O}(G)$ is naturally embedded into the Hopf dual $\mathcal{U}(\mathfrak{g})^o$ and similarly $\overline{\mathcal{O}(G)} \subset \mathcal{U}(\overline{\mathfrak{g}})^o$. In our previous discussion we proved the existence of $\sigma' :$

$\overline{\mathcal{O}(G)} \longrightarrow \mathcal{O}(G)$ inducing \hat{s} , consequently such σ' must extend to give the morphism $\mathcal{U}(\hat{s})^\circ : \mathcal{U}(\mathfrak{g})^\circ \longrightarrow \mathcal{U}(\overline{\mathfrak{g}})^\circ$ and this proves the uniqueness.

We are ready to define the compact form of our complex supergroup G .

Definition 5.5. We define the K , *compact form* of G , as the algebraic supergroup represented by the Hopf superalgebra $\mathcal{O}(G)^{\sigma'}$ (where σ' is as in Observation 5.4):

$$K : (\text{salg})_{\mathbf{R}} \longrightarrow (\text{sets}), \quad K(A) = \text{Hom}_{(\text{salg})_{\mathbf{R}}}(\mathcal{O}(G)^{\sigma'}, A)$$

Notice that $\mathcal{O}(G)^{\sigma'}$ is a real form of $\mathcal{O}(G)$.

The next proposition allows us to calculate the functor of points of the supergroup K .

Proposition 5.6. *Let the notation be as above. Then $K(A)$ coincides with the σ_A invariants in $G^r(A)$:*

$$K(A) = G^r(A)^{\sigma_A} \subset G^r(A)$$

Furthermore $\text{Lie}(K) = \mathfrak{k}$.

Proof. The first statement is a consequence of our definitions. In fact since $\mathcal{O}(G) = \mathcal{O}(G)^\sigma \otimes_{\mathbf{R}} \mathbf{C}$,

$$\text{Hom}_{(\text{salg})_{\mathbf{R}}}(\mathcal{O}(G)^{\sigma'}, A \otimes_{\mathbf{R}} \mathbf{C}) = G_V(A \otimes_{\mathbf{R}} \mathbf{C}) = G^r(A)$$

In Obs. 5.4 we established that $\sigma_A = c \circ \underline{\text{Spec}}(\sigma')$. Notice that σ' is the identity on $\mathcal{O}(G)^{\sigma'}$ and c is the automorphism induced by complex conjugation. Hence $K(A) = G^r(A)^{\sigma_A}$. As for the Lie superalgebras statement, it is enough to notice that $\mathcal{U}(s) : \mathcal{U}(\mathfrak{g}) \longrightarrow \mathcal{U}(\mathfrak{g})$ corresponds to $\mathcal{U}(\hat{s})$ as in Obs. 5.4, the unique extension of $\sigma' : \mathcal{O}(G) \longrightarrow \mathcal{O}(G)$ to $\mathcal{U}(\mathfrak{g})^\circ$. Hence the fixed points of σ' correspond to $\mathcal{U}(\mathfrak{g})^{\mathcal{U}(s)} = \mathcal{U}(\mathfrak{k})$. \blacksquare

We now consider few examples.

Example 5.7. 1. $\text{SL}_2(\mathbf{C})^r$. Let us write the involution σ on the big cell of $\text{SL}_2(\mathbf{C})^r$, that is on U^-TU^+ where U^\pm denote the lower and upper unipotent subgroups, and T the diagonal torus. Such σ will extend uniquely to the whole $\text{SL}_2(\mathbf{C})^r$. The big cell inside $\text{SL}_2(A)^r := \text{SL}_2(\mathbf{C})^r(A)$ is

$$\begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} t & tu \\ vt & utv + t^{-1} \end{pmatrix}$$

Let's apply σ :

$$\sigma \left(\begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right) = \begin{pmatrix} 1 & -\bar{v} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{t}^{-1} & 0 \\ 0 & \bar{t} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\bar{u} & 1 \end{pmatrix} =$$

$$\begin{pmatrix} \bar{t}^{-1} + \bar{u}\bar{t}\bar{v} & -\bar{v}\bar{t} \\ -\bar{t}\bar{u} & \bar{t} \end{pmatrix}$$

Hence the condition to impose is:

$$\begin{pmatrix} t & tv \\ tu & utv + t^{-1} \end{pmatrix} = \begin{pmatrix} \bar{t}^{-1} + \bar{u}\bar{t}\bar{v} & -\bar{u}\bar{t} \\ -\bar{t}\bar{u} & \bar{t} \end{pmatrix}$$

Since any condition on the big cell and local real (super)algebras will extend uniquely to $\mathrm{SL}_2(A)^r$ we have:

$$K(A) := (\mathrm{SL}_2(A)^r)^\sigma = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \mid |a|^2 + |b|^2 = 1 \right\}$$

which is $SU(2)$ as expected.

2. $\mathrm{GL}(1|1)$. For $\mathrm{GL}(1|1)$ we have global coordinates and we can write the generic element g of $\mathrm{GL}(1|1)(A)$ as:

$$g = \begin{pmatrix} 1 & 0 \\ \theta & 1 \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & s \end{pmatrix} \begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} t & t\eta \\ t\theta & \theta t\eta + s \end{pmatrix} = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix}$$

Applying σ we get:

$$\sigma(g) = \begin{pmatrix} 1 & -\bar{\theta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{t}^{-1} & 0 \\ 0 & \bar{s}^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\bar{\eta} & 1 \end{pmatrix} = \begin{pmatrix} \bar{t}^{-1} + \bar{\theta}\bar{s}^{-1}\bar{\eta} & -\bar{s}^{-1}\bar{\theta} \\ -\bar{s}^{-1}\bar{\eta} & \bar{s}^{-1} \end{pmatrix}$$

Hence the condition to impose is:

$$\begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix} := \begin{pmatrix} t & t\eta \\ t\theta & \theta t\eta + t \end{pmatrix} = \begin{pmatrix} \bar{t}^{-1} + \bar{\theta}\bar{s}^{-1}\bar{\eta} & -\bar{s}^{-1}\bar{\theta} \\ -\bar{s}^{-1}\bar{\eta} & \bar{s}^{-1} \end{pmatrix}$$

which gives after some straightforward calculations:

$$K(A) = (\mathrm{GL}(1|1)(A)^r)^\sigma = \left\{ \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix} \mid a = \bar{a}^{-1} + \beta d^{-1} \gamma, d^{-1} = \bar{d} + \bar{\beta} \bar{a}^{-1} \bar{\gamma}, \right.$$

$$\left. \beta = -\bar{a}^{-1} \bar{\gamma} d, \gamma = -d \bar{\beta} \bar{a}^{-1} \right\}$$

3. $SL(1|1)$. For $SL(1|1)$ we have global coordinates and we can write the generic element g of $SL(1|1)(A)$ as:

$$g = \begin{pmatrix} 1 & 0 \\ \theta & 1 \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} \begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} t & t\eta \\ t\theta & \theta t\eta + t \end{pmatrix}$$

Applying σ we get:

$$\sigma(g) = \begin{pmatrix} 1 & -\bar{\theta} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \bar{t}^{-1} & 0 \\ 0 & \bar{t}^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\bar{\eta} & 1 \end{pmatrix} = \begin{pmatrix} \bar{t}^{-1} + \bar{\theta}\bar{t}^{-1}\bar{\eta} & -\bar{t}^{-1}\bar{\theta} \\ -\bar{t}^{-1}\bar{\eta} & \bar{t}^{-1} \end{pmatrix}$$

Hence the condition to impose is:

$$\begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix} := \begin{pmatrix} t & t\eta \\ t\theta & \theta t\eta + t \end{pmatrix} = \begin{pmatrix} \bar{t}^{-1} + \bar{\theta}\bar{t}^{-1}\bar{\eta} & -\bar{t}^{-1}\bar{\theta} \\ -\bar{t}^{-1}\bar{\eta} & \bar{t}^{-1} \end{pmatrix}$$

which gives in terms of the entries a, d, β, γ :

$$K(A) = (SL(1|1)(A)^r)^\sigma =: SU(1|1) = \left\{ \begin{pmatrix} a & \beta \\ -\bar{\beta} & \bar{a}^{-1} \end{pmatrix} \mid \bar{a}(a + \beta\bar{a}\bar{\beta}) = 1 \right\}$$

Notice that the reduced part is a torus as one expects.

4. $SL(m|n)$. The calculation resembles very much (2), this time the letters are matrices.

$$\begin{aligned} \sigma \left(\begin{pmatrix} 1 & 0 \\ \theta & 1 \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & s \end{pmatrix} \begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix} \right) &= \begin{pmatrix} 1 & -\bar{\theta}^t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} (\bar{t}^{-1})^t & 0 \\ 0 & (\bar{s}^{-1})^t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\bar{\eta}^t & 1 \end{pmatrix} = \\ &= \begin{pmatrix} (\bar{t}^{-1})^t + \bar{\theta}^t(\bar{s}^{-1})^t\bar{\eta}^t & -\theta^t(\bar{s}^{-1})^t \\ -(\bar{s}^{-1})^t\bar{\eta}^t & (\bar{s}^{-1})^t \end{pmatrix} \end{aligned}$$

The condition to impose is:

$$\begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix} := \begin{pmatrix} t & t\eta \\ \theta t & \theta t\eta + s \end{pmatrix} = \begin{pmatrix} (\bar{t}^{-1})^t + \bar{\theta}^t(\bar{s}^{-1})^t\bar{\eta}^t & -\theta^t(\bar{s}^{-1})^t \\ -(\bar{s}^{-1})^t\bar{\eta}^t & (\bar{s}^{-1})^t \end{pmatrix}$$

which gives:

$$K(A) = (SL(m|n)(A)^r)^\sigma =: SU(m|n) = \left\{ \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix} \right\}$$

with

$$a = (\bar{a}^{-1})^t + \beta d^{-1}\gamma, \quad \beta = -(\bar{a}^{-1})^t\bar{\gamma}^t d$$

$$\gamma = -d\bar{\beta}^t(\bar{a}^{-1})^t \quad d = (\bar{d}^{-1})^t + \gamma a^{-1}\beta$$

6 The SHCP's approach to compact forms

We recall that we have a SHCP when we are given a pair (G_0, \mathfrak{g}) consisting of an ordinary Lie group and a Lie superalgebra, such that $\text{Lie}(G_0) = \mathfrak{g}_0$ and G_0 acts on \mathfrak{g} in such a way that the differential of the action is the Lie superbracket in \mathfrak{g} (Ref. [13]). The category of SHCP's is equivalent to the category of super Lie groups. In [4] such equivalence is extended to the categories of analytic and algebraic supergroups, with the obvious changes in the definition of SHCP (see also [14] for a more comprehensive treatment of this equivalence).

If G is a complex (analytic or algebraic) supergroup with Lie superalgebra of classical type and with G_0 the reduced underlying ordinary group, (G_0, \mathfrak{g}) is a SHCP, where $\mathfrak{g} = \text{Lie}(G)$.

We now turn to our setting. Let G be a complex analytic Lie supergroup, with Lie superalgebra \mathfrak{g} of classical type, subject to hypothesis (1). Let $K_0 \subset G_0$ be the maximal compact Lie subgroup of G_0 and \mathfrak{k} as in 4.1. Then $K_{shcp} = (K_0, \mathfrak{k})$ is a SHCP. In fact by the ordinary theory we have that $\mathfrak{k}_0 = \text{Lie}(K_0)$, moreover the natural adjoint action of K_0 on \mathfrak{k} obtained by restricting the adjoint action of G_0 on \mathfrak{g} induces the bracket.

Definition 6.1. Let the notation and assumptions be as above. We define $K_{shcp} = (K_0, \mathfrak{k})$ the *shcp maximal compact* in G .

The supergroup K_{shcp} corresponds, via the above mentioned equivalence of categories, to the maximal compact subgroup K as defined in the previous sections. In fact they both have the same underlying Lie group K_0 , the same Lie superalgebra \mathfrak{k} and K_0 acts on \mathfrak{k} in the same way in both cases. We now want to give a more stringent characterization of the superalgebra of the global sections on K so to establish a link between the SHCP's and the functor of points approach detailed in the previous section.

We recall that if (G_0, \mathfrak{g}) is a SHCP, we can obtain the global sections of the corresponding algebraic supergroup G as follows:

$$\mathcal{O}(G_0, \mathfrak{g}) = \text{Hom}_{\mathcal{U}(\mathfrak{g}_0)}(\mathcal{U}(\mathfrak{g}), \mathcal{O}(G_0))$$

(for more details on these definitions see [5] ch. 7, sec. 7.4). The equivalence of the category of SHCP and algebraic supergroups prescribes that

$\mathcal{O}(G_0, \mathfrak{g}) \cong \mathcal{O}(G)$ through the isomorphism:

$$\begin{aligned} \mathcal{O}(G) &\longrightarrow \mathcal{O}(G_0, \mathfrak{g}) \\ s &\longmapsto X \mapsto (-1)^{|X|} |D_X s| \end{aligned}$$

where D_X is the left invariant differential operator associated with $X \in \mathcal{U}(\mathfrak{g})$.

Let s be the involution of the complex simple Lie superalgebra \mathfrak{g} (as in 4.2) such that $\mathfrak{g}^s = \mathfrak{k}$. The involution s induces an involution on the universal enveloping algebra (still denoted with s) and we have $\mathcal{U}(\mathfrak{g})^s = \mathcal{U}(\mathfrak{k})$.

We define the following subalgebra of $\mathcal{O}(G_0, \mathfrak{g})$:

$$\mathcal{O}(G_0, \mathfrak{g})^{s, \sigma_0} = \{f : \mathcal{U}(\mathfrak{g}) \longrightarrow \mathcal{O}(G_0) \mid f \cdot s = f, \sigma_0 \cdot f(u) = f(u)\} \subset \mathcal{O}(G_0, \mathfrak{g})$$

Proposition 6.2. *Let the notation be as above. Then:*

$$\mathcal{O}(K_0, \mathfrak{k}) = \mathcal{O}(G_0, \mathfrak{g})^{s, \sigma_0}.$$

Proof. Define

$$\begin{aligned} \psi : \mathcal{O}(G_0, \mathfrak{g})^{s, \sigma_0} &\longrightarrow \mathcal{O}(K_0, \mathfrak{k}) \\ f &\longmapsto f|_{\mathcal{U}(\mathfrak{k})} \end{aligned}$$

Since $f(u) = \sigma_0 \cdot f(u)$ we have that ψ is well defined, i.e. $f(u) \in \mathcal{O}(K_0) = \mathcal{O}(G_0)^{\sigma_0}$. Furthermore ψ is a superalgebra morphism. Now the fact ψ is injective. Assume $f|_{\mathcal{U}(\mathfrak{k})} = 0$. Since $f \circ s = f$ we have $f(s(u)) = f(u)$. $u + s(u)$ is invariant, hence $f(u + s(u)) = 2f(u) = 0$ for all $u \in \mathcal{U}(\mathfrak{g})$. Now the surjectivity. Let $g \in \mathcal{O}(K_0, \mathfrak{k})$, we want to determine $f \in \mathcal{O}(G_0, \mathfrak{g})^{s, \sigma_0}$ so that $f|_{\mathcal{U}(\mathfrak{k})} = g$. If such f exists, it must coincide with g on the s invariant elements of $\mathcal{U}(\mathfrak{g})$. Hence: $g(s(u) + u) = f(s(u) + u) = 2f(u)$. Hence we define: $f = (1/2)g(s(u) + u)$. \blacksquare

The next corollary establishes a direct link between the two approaches to the definition of K maximal compact in G .

Corollary 6.3. *Let the notation be as above. Then*

$$\mathcal{O}(K) = \mathcal{O}(G)^{\sigma'} \cong \mathcal{O}(G_0, \mathfrak{g})^{s, \sigma_0} = \mathcal{O}(K_0, \mathfrak{k}) = \mathcal{O}(K_{shcp})$$

in other words we have an explicit isomorphism between the superalgebras of global sections of the maximal compact subgroup K of G defined as in 5.5 and the maximal compact subgroup K_{shcp} corresponding to the SHCP (K_0, \mathfrak{k}) .

References

- [1] Alldridge A., *The Harish-Chandra isomorphism for reductive symmetric superpairs*, arXiv:1004.0732, 2010.
- [2] Balduzzi L., Carmeli C., Cassinelli G., *Representations of super Lie groups: some remarks*, "Supersymmetry in Mathematics and Physics" Eds. S. Ferrara, R. Fiorese, V.S.Varadarajan, Springer LNM 2027, 2011.
- [3] A. Borel, *Linear Algebraic Groups*, Springer Verlag GTE.
- [4] C. Carmeli, R. Fiorese, *Super Distributions, Analytic and Algebraic Super Harish-Chandra Pairs* preprint, arxiv.org/pdf/1106.1072, 2011.
- [5] C. Carmeli, L. Caston, R. Fiorese, *Mathematical Foundation of Supersymmetry*, with an appendix with I. Dimitrov, EMS Ser. Lect. Math., European Math. Soc., Zurich, 2011.
- [6] P. Deligne, J. Morgan, *Notes on supersymmetry (following J. Bernstein)*, in: "Quantum fields and strings. A course for mathematicians", Vol. 1, AMS, 1999.
- [7] M. Demazure, A. Grothendieck, *Schémas en groupes, III*, Séminaire de Géométrie Algébrique du Bois Marie, Vol. 3, 1964.
- [8] R. Fiorese, *Smoothness of algebraic supervarieties and supergroups*, Pac. J. Math., 234, pp. 295 - 310, 2008.
- [9] R. Fiorese, F. Gavarini, *Chevalley Supergroups*, Memoirs of the AMS, **215**, 2012.
- [10] R. Fiorese, F. Gavarini, *On Algebraic Supergroups with Lie superalgebras of classical type*, arXiv:1106.4168v3 preprint, 2011.
- [11] R. Hartshorne. *Algebraic geometry*. Graduate Text In Mathematics. Springer-Verlag, New York, 1977.
- [12] V. G. Kac, *Lie superalgebras*, Adv. Math. **26** (1977), 8–26.
- [13] J.-L. Koszul, *Graded manifolds and graded Lie algebras*, Proceedings of the international meeting on geometry and physics (Florence, 1982), 71–84, Pitagora, Bologna, 1982.

- [14] A. Masuoka, *Harish-Chandra pairs for algebraic affine supergroup schemes over an arbitrary field*, arXiv:1111.2387, 2011.
- [15] R. Steinberg, *Lectures on Chevalley groups*, Yale University, New Haven, Conn., 1968.
- [16] V. S. Varadarajan, *Supersymmetry for mathematicians: an introduction*, Courant Lecture Notes **1**, AMS, 2004.